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Relationship between various supersymmetric lattice models

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Abstract: We comment on the relationships between several supersymmetric lattice models; the "orbifold lattice theory" by Cohen-Kaplan-Katz-Unsal (CKKU), lattice regularization of the topological field theory by Sugino and the "geometrical approach" by Catterall. We point out that these three models have close relationships; the $\mathcal{N} = (2,2)$ model by Catterall [1] and the two-dimensional $\mathcal{N} = (2,2)$ lattice theory being similar to Sugino's construction [2] can be derived by appropriate truncation of fields in the two-dimensional $\mathcal{N} = (4,4)$ orbifold lattice theory by CKKU [3]. Catterall's $\mathcal{N} = (2,2)$ description possesses extra degrees of freedom compared to the target $\mathcal{N} = (2,2)$ theory. If we remove those extra degrees of freedom in a way keeping supersymmetry on the lattice, Catterall's description reduces to a model of the Sugino type.

Keywords: Topological Field Theories, Lattice Gauge Field Theories, Extended Supersymmetry.

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Contents

1. Introduction

Recently, several lattice gauge theories which preserve partial supersymmetry on the lattice are proposed [1 − 14].[1] The main purpose in these models is to solve the fine-tuning problem in lattice regularizations of supersymmetric gauge theories. The models utilize the topological twisting [24, 25] to pick up the subset of superalgebra which does not include the infinitesimal translation. In this way, the partial supersymmetry can be preserved on the lattice which explicitly breaks the infinitesimal translational invariance.

[1] There are other several supersymmetric lattice models which are not treated in this paper [15 − 23].

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There are several types of the model: The series of models proposed by Cohen-Kaplan-Katz-Unsal-Endres [3 − 7] are "orbifold lattices" which are constructed from reduced supersymmetric matrix models by the orbifold projection [26] and the deconstruction [27]. In their way, the orbifold projection generates the lattice theory with preserved subset of supersymmetry of the target theory. The deconstruction dynamically generates space-time by the vacuum expectation value $\frac{1}{\sqrt{2}a}$ of bosonic link fields, where a denotes the lattice spacing.

The other approach, proposed by Sugino [2, 9 − 12], are lattice regularizations of the "topological field theory action" which is equivalent to the extended supersymmetric gauge theory. In his approach, the BRST-like supercharges are preserved on the lattice because such charges do not generate the infinitesimal translation.

Catterall proposed models [1, 13, 14] which are based on the Kahler-Dirac formalism and the lattice analogue of differential forms [28]. In his models, the 1-form and 2-form fields have to be complex because they are in the bi-fundamental representation of the lattice gauge group and the hermiticity cannot be maintained under gauge transformations. Since the counterparts of these 1- and 2-form fields in the target theory are hermitian, Catterall's models have extra degrees of freedom which we have to discard in the path-integral. If one performs such truncation in a naive way, supersymmetry on the lattice would be broken.

Seemingly, these three types of model are quite different. There exist, however, close relationships between them. We will clarify such relationships in this paper. This investigation of the relationships would be very useful to develop the lattice formulations of supersymmetric theories. First, in section 2, we show that Catterall's $\mathcal{N} = (2, 2)$ action [1] can be embedded in CKKU's $\mathcal{N} = (4, 4)$ action [3] under suitable field truncation. Then, in section 3, we explain the relationship between Catterall's "complexified" $\mathcal{N} = (2, 2)$ lattice theory and Sugino's theory of ref. [2]. For Catterall's model to contain the correct numbers of degrees of freedom compared to the target $\mathcal{N} = (2, 2)$ theory, we have to truncate extra degrees of freedom. If we perform the truncation in a way keeping the supersymmetry on the lattice, Catterall's model becomes the $\mathcal{N} = (2, 2)$ model being similar to Sugino's model. Finally, in section 4, we explain that the $\mathcal{N} = (2, 2)$ supersymmetric lattice model of the Sugino type can be directly derived from CKKU's $\mathcal{N} = (4, 4)$ lattice theory by restricting fields. We also explain that the derivation discards the quantum fluctuations of scalar zero modes around the vacuum expectation value $\frac{1}{\sqrt{2}a}$. In section 5, we also give a continuum analogue of the truncation of degrees of freedom. By this truncation, we can obtain the continuum $\mathcal{N} = (2, 2)$ super Yang-Mills theory from the continuum $\mathcal{N} = (4, 4)$ super Yang-Mills theory. Section 6 is devoted to conclusion and discussion.

2. Relationship between the $\mathcal{N} = (4, 4)$ CKKU model and the $\mathcal{N} = (2, 2)$ Catterall model

Here we explicitly show that Catterall's $\mathcal{N} = (2, 2)$ lattice model can be obtained by truncating certain fields in the $\mathcal{N} = (4, 4)$ CKKU model.

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2.1 $\mathcal{Q}$-exact form of the $\mathcal{N} = (4,4)$ CKKU model

We explain the $\mathcal{N} = (4,4)$ supersymmetric lattice theory proposed by CKKU in ref. [3] very briefly. Here we neglect the soft supersymmetry breaking mass term in their theory.[2] The action of the theory (eq. (3.14) of ref. [3]) is

$$
\begin{aligned}
S = \sum_{\mathbf{n}} \mathrm{Tr}\Bigg[&\int d\theta d\bar\theta \left(\frac{1}{2}\overline{\mathbf{\Upsilon}}_{\mathbf{n}}\mathbf{\Upsilon}_{\mathbf{n}} + \frac{1}{\sqrt{2}}\mathbf{S}_{\mathbf{n}}(\mathbf{Z}_{i,\mathbf{n}}\overline{\mathbf{Z}}_{i,\mathbf{n}} - \overline{\mathbf{Z}}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}\mathbf{Z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}) - \frac{1}{2}\overline{\mathbf{\Xi}}_{\mathbf{n}}\mathbf{\Xi}_{\mathbf{n}} \right) \\
&+ \int d\theta \left(\epsilon_{ij}\, \mathbf{\Xi}_{\mathbf{n}}\, \mathbf{Z}_{i,\mathbf{n}}\mathbf{Z}_{j,\mathbf{n}+\hat{\mathbf{e}}_i} \right) - \int d\bar\theta \left(\epsilon_{ij}\, \overline{\mathbf{\Xi}}_{\mathbf{n}}\, \overline{\mathbf{Z}}_{i,\mathbf{n}+\hat{\mathbf{e}}_j}\overline{\mathbf{Z}}_{j,\mathbf{n}} \right) \Bigg],
\end{aligned}
\tag{2.1}
$$

where the sum over site $\mathbf{n} = \{n_1, n_2\}$ is taken in the interval $n_{1,2} \in [1, N]$ and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ being unit vectors in n_1 and n_2 directions, respectively. The superfields are defined by

$$
\begin{aligned}
\mathbf{Z}_{i,\mathbf{n}} &= z_{i,\mathbf{n}} + \sqrt{2}\,\theta\psi_{i,\mathbf{n}} - \sqrt{2}\,\theta\bar\theta(\bar z_{3,\mathbf{n}} z_{i,\mathbf{n}} - z_{i,\mathbf{n}}\bar z_{3,\mathbf{n}+\hat{\mathbf{e}}_i}), \\
\overline{\mathbf{Z}}_{i,\mathbf{n}} &= \bar z_{i,\mathbf{n}} + \sqrt{2}\,\bar\theta\epsilon_{ij}\xi_{j,\mathbf{n}} + \sqrt{2}\,\theta\bar\theta(\bar z_{3,\mathbf{n}+\hat{\mathbf{e}}_i}\bar z_{i,\mathbf{n}} - \bar z_{i,\mathbf{n}}\bar z_{3,\mathbf{n}}), \\
\mathbf{\Xi}_{\mathbf{n}} &= \xi_{3,\mathbf{n}} + \sqrt{2}\,\theta\tilde G_{\mathbf{n}} - \sqrt{2}\,\theta\bar\theta(\bar z_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}\xi_{3,\mathbf{n}} - \bar z_{3,\mathbf{n}}\xi_{3,\mathbf{n}}), \\
\overline{\mathbf{\Xi}}_{\mathbf{n}} &= \chi_{\mathbf{n}} - \sqrt{2}\,\bar\theta\tilde{\bar G}_{\mathbf{n}} + \sqrt{2}\,\theta\bar\theta(\bar z_{3,\mathbf{n}}\chi_{\mathbf{n}} - \chi_{\mathbf{n}}\bar z_{\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}), \\
\mathbf{S}_{\mathbf{n}} &= z_{3,\mathbf{n}} + \sqrt{2}\,\theta\psi_{3,\mathbf{n}} + \sqrt{2}\,\bar\theta\lambda_{\mathbf{n}} + \sqrt{2}\,\theta\bar\theta i\tilde d_{\mathbf{n}}, \\
\mathbf{\Upsilon}_{\mathbf{n}} &= \lambda_{\mathbf{n}} - \theta\Big(i\tilde d_{\mathbf{n}} + [\bar z_{3,\mathbf{n}}, z_{3,\mathbf{n}}]+\Big) - \sqrt{2}\theta\bar\theta[\bar z_{3,\mathbf{n}}, \lambda_{\mathbf{n}}], \\
\overline{\mathbf{\Upsilon}}_{\mathbf{n}} &= \psi_{3,\mathbf{n}} + \bar\theta\Big(i\tilde d_{\mathbf{n}} - [\bar z_{3,\mathbf{n}}, z_{3,\mathbf{n}}]\Big) + \sqrt{2}\theta\bar\theta[\bar z_{3,\mathbf{n}}, \psi_{3,\mathbf{n}}],
\end{aligned}
\tag{2.2}
$$

with

$$
\begin{aligned}
\tilde{\bar G}_{\mathbf{n}} &= \bar G_{\mathbf{n}} - \sqrt{2}\,\epsilon_{ij}\, z_{i,\mathbf{n}} z_{j,\mathbf{n}+\hat{\mathbf{e}}_i}, \\
\tilde G_{\mathbf{n}} &= G_{\mathbf{n}} - \sqrt{2}\,\epsilon_{ij}\, \bar z_{i,\mathbf{n}+\hat{\mathbf{e}}_j}\bar z_{j,\mathbf{n}}, \\
\tilde d_{\mathbf{n}} &= d_{\mathbf{n}} - i(\bar z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - z_{i,\mathbf{n}}\bar z_{i,\mathbf{n}}).
\end{aligned}
\tag{2.3}
$$

In these expressions, θ, $\bar\theta$ are one-component Grassmann super coordinates. All variables are $M \times M$ matrices satisfying periodic boundary conditions on the lattice, and there is an independent $\mathrm{U}(M)$ symmetry associated with each site which becomes the $\mathrm{U}(M)$ gauge symmetry of the continuum theory. The indices i, j run over 1 and 2 and all repeated indices are summed. The variables z_a ($a = 1,2,3$) and $\bar z_a$ refer to complex bosonic variables and their conjugates, while λ, χ, ψ_a and ξ_a refer to one-component Grassmann variables. Here $d_{\mathbf{n}}$, $G_{\mathbf{n}}$, $\bar G_{\mathbf{n}}$ are auxiliary fields originally introduced in ref. [3]. These auxiliary fields are integrated out yielding $d_{\mathbf{n}} = G_{\mathbf{n}} = \bar G_{\mathbf{n}} = 0$.

[2]In CKKU's models, there are flat directions in the scalar potential allowing large fluctuations around the vacuum expectation value $\frac{1}{\sqrt{2}a}$. To stabilize the lattice structure, CKKU introduced the soft SUSY breaking mass terms. In this section, we will investigate their model without such mass terms.

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The supersymmetry on the lattice can be read off from eq. (2.2). It is

$$
\begin{aligned}
\delta z_{i,\mathbf{n}} &= i\sqrt{2}\,\eta\,\psi_{i,\mathbf{n}},\\
\delta \overline{z}_{i,\mathbf{n}} &= i\epsilon_{ij}\,\sqrt{2}\,\overline{\eta}\,\xi_{j,\mathbf{n}},\\
\delta \psi_{i,\mathbf{n}} &= 2i\overline{\eta}\,[z_{i,\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_i} - \overline{z}_{3,\mathbf{n}}z_{i,\mathbf{n}}],\\
\delta \xi_{i,\mathbf{n}} &= -2i\epsilon_{ij}\,\eta\,[\overline{z}_{j,\mathbf{n}}\overline{z}_{3,\mathbf{n}} - \overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_j}\overline{z}_{j,\mathbf{n}}],\\
\delta z_{3,\mathbf{n}} &= i\sqrt{2}\,(\eta\,\psi_{3,\mathbf{n}} + \overline{\eta}\,\lambda_{\mathbf{n}}),\\
\delta \overline{z}_{3,\mathbf{n}} &= 0,\\
\delta \psi_{3,\mathbf{n}} &= i\overline{\eta}\,\left([\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - z_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}}] - [\overline{z}_{3,\mathbf{n}},\, z_{3,\mathbf{n}}] + id_{\mathbf{n}}\right),\\
\delta \lambda_{\mathbf{n}} &= -i\eta\,\left([\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - z_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}}] + [\overline{z}_{3,\mathbf{n}},\, z_{3,\mathbf{n}}] + id_{\mathbf{n}}\right),\\
\delta \chi_{\mathbf{n}} &= i\overline{\eta}\,[2(z_{1,\mathbf{n}}z_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - z_{2,\mathbf{n}}z_{1,\mathbf{n}+\hat{\mathbf{e}}_2}) - \sqrt{2}\ \overline{G}_{\mathbf{n}}],\\
\delta \xi_{3,\mathbf{n}} &= -i\eta\,[2(\overline{z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2}\overline{z}_{2,\mathbf{n}} - \overline{z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\overline{z}_{1,\mathbf{n}}) - \sqrt{2}G_{\mathbf{n}}],
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
\delta\overline{G}_{\mathbf{n}} &= 2i\eta\,\left(\epsilon_{ij}\,(z_{i,\mathbf{n}}\psi_{j,\mathbf{n}+\hat{\mathbf{e}}_i} - \psi_{j,\mathbf{n}}z_{i,\mathbf{n}+\hat{\mathbf{e}}_j}) + (\overline{z}_{3,\mathbf{n}}\chi_{\mathbf{n}} - \chi_{\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2})\right),\\
\delta G_{\mathbf{n}} &= -2i\overline{\eta}\left(\sum_{i,j,\,\mathrm{with}\,i\neq j}(\overline{z}_{i,\mathbf{n}+\hat{\mathbf{e}}_j}\xi_{i,\mathbf{n}} - \xi_{i,\mathbf{n}+\hat{\mathbf{e}}_i}\overline{z}_{i,\mathbf{n}}) + (\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}\xi_{3,\mathbf{n}} - \xi_{3,\mathbf{n}}\overline{z}_{3,\mathbf{n}})\right),\\
\delta d_{\mathbf{n}} &= -\sqrt{2}\eta(\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}\psi_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - \psi_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}} + [\overline{z}_{3,\mathbf{n}},\psi_{3,\mathbf{n}}])\\
&\quad + \sqrt{2}\overline{\eta}(\epsilon_{ij}(z_{i,\mathbf{n}}\xi_{j,\mathbf{n}} - \xi_{j,\mathbf{n}-\hat{\mathbf{e}}_i}z_{i,\mathbf{n}-\hat{\mathbf{e}}_i}) + [\overline{z}_{3,\mathbf{n}},\lambda_{\mathbf{n}}]).
\end{aligned}
\tag{2.5}
$$

We may express the supersymmetry transformation by using two supercharges

$$
\delta = i\eta Q + i\overline{\eta}\overline{Q}. \tag{2.6}
$$

These charges Q, $\overline{Q}$ can be realized in terms of independent Grassmann coordinates θ and $\overline{\theta}$ as

$$
Q = \frac{\partial}{\partial\theta} + \sqrt{2}\,\overline{\theta}[\overline{z}_3,\,\cdot\,]^*, \qquad \overline{Q} = \frac{\partial}{\partial\overline{\theta}} + \sqrt{2}\,\theta[\overline{z}_3,\,\cdot\,]^*, \tag{2.7}
$$

where the operation $[\overline{z}_3,\,\cdot\,]^*$ represents the lattice gauge transformation with the parameter $\overline{z}_3$. This operation $[\overline{z}_3,\,\cdot\,]^*$ acts on generic fields $P_{\mathbf{n}}$ living on the links as

$$
[\overline{z}_3,\, P_{\mathbf{n}}]^* \equiv \overline{z}_{3,\mathbf{n}}P_{\mathbf{n}} - P_{\mathbf{n}}\overline{z}_{3,\mathbf{n}+r_i\hat{\mathbf{e}}_i}, \tag{2.8}
$$

where we have assumed that the link under consideration connects two sites $\mathbf{n}$ and $\mathbf{n}+r_i\hat{\mathbf{e}}_i$. This rule is applied to $z_{i,\mathbf{n}}$, $\psi_{i,\mathbf{n}}$, $\chi_{\mathbf{n}}$ and $\overline{G}_{\mathbf{n}}$. Similarly, for the anti-oriented link fields $\overline{P}_{\mathbf{n}}$, such as $\overline{z}_{i,\mathbf{n}}$, $\xi_{i,\mathbf{n}}$, $\xi_{3,\mathbf{n}}$ and $G_{\mathbf{n}}$,

$$
[\overline{z}_3,\, \overline{P}_{\mathbf{n}}]^* \equiv \overline{z}_{3,\mathbf{n}+r_i\hat{\mathbf{e}}_i}\overline{P}_{\mathbf{n}} - \overline{P}_{\mathbf{n}}\overline{z}_{3,\mathbf{n}}. \tag{2.9}
$$

For site fields $P'_{\mathbf{n}}$, which are $z_{3,\mathbf{n}}$, $\lambda_{\mathbf{n}}$, $\psi_{3,\mathbf{n}}$ and $d_{\mathbf{n}}$, the operation is simply the commutator $[\overline{z}_{3,\mathbf{n}}, P'_{\mathbf{n}}]^* \equiv [\overline{z}_{3,\mathbf{n}}, P'_{\mathbf{n}}]$. Auxiliary fields $G_{\mathbf{n}}$, $\overline{G}_{\mathbf{n}}$, $d_{\mathbf{n}}$ and their transformation laws (2.5) are

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introduced to make the algebra,[3] $Q^2 = \overline{Q}^2 = 0$ and

$$\{Q, \overline{Q}\} = -2\sqrt{2}[\overline{z}_3, \cdot]^*, \tag{2.10}$$

to hold off-shell.

We define the BRST-like charge $\mathcal{Q}$ by

$$\mathcal{Q} = \frac{1}{\sqrt{2}}\left(Q - \overline{Q}\right), \tag{2.11}$$

which induces

$$\begin{aligned}
&\mathcal{Q} z_{i,\mathbf{n}} = \psi_{i,\mathbf{n}}, && \mathcal{Q}\psi_{i,\mathbf{n}} = \sqrt{2}(\overline{z}_{3,\mathbf{n}} z_{i,\mathbf{n}} - z_{i,\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_i}),\\
&\mathcal{Q}\overline{z}_{i,\mathbf{n}} = -\epsilon_{ij}\xi_{j,\mathbf{n}}, && \mathcal{Q}\xi_{i,\mathbf{n}} = \sqrt{2}\epsilon_{ij}(\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_j}\overline{z}_{j,\mathbf{n}} - \overline{z}_{j,\mathbf{n}}\overline{z}_{3,\mathbf{n}}),\\
&\mathcal{Q} z_{3,\mathbf{n}} = \psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}, && \mathcal{Q}(\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}) = \sqrt{2}[\overline{z}_{3,\mathbf{n}}, z_{\mathbf{n}}],\\
&\mathcal{Q}\tilde{d}_{\mathbf{n}} = i[\overline{z}_{3,\mathbf{n}}, \psi_{3,\mathbf{n}} + \lambda_{\mathbf{n}}], && \mathcal{Q}(\psi_{3,\mathbf{n}} + \lambda_{\mathbf{n}}) = -\sqrt{2}i\tilde{d}_{\mathbf{n}},\\
&\mathcal{Q}\chi_{\mathbf{n}} = \tilde{\overline{G}}_{\mathbf{n}}, && \mathcal{Q}\tilde{\overline{G}}_{\mathbf{n}} = \sqrt{2}(\overline{z}_{3,\mathbf{n}}\chi_{\mathbf{n}} - \chi_{\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}),\\
&\mathcal{Q}\xi_{3,\mathbf{n}} = \tilde{G}_{\mathbf{n}}, && \mathcal{Q}\tilde{G}_{\mathbf{n}} = \sqrt{2}(\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}\xi_{3,\mathbf{n}} - \xi_{3,\mathbf{n}}\overline{z}_{3,\mathbf{n}}),\\
&\mathcal{Q}\overline{z}_{3,\mathbf{n}} = 0.
\end{aligned} \tag{2.12}$$

This charge $\mathcal{Q}$ also satisfies

$$\mathcal{Q}^2 = \sqrt{2}[\overline{z}_3, \cdot]^*, \tag{2.13}$$

where the right hand side is the gauge transformation with the parameter $\overline{z}_3$.

Now, for our purpose, it is crucial to rewrite the action (2.1) in a $\mathcal{Q}$-exact form. Then it can be confirmed that the action is $\mathcal{Q}$-exact

$$S = \frac{1}{2g^2}\mathcal{Q}\,\Xi, \tag{2.14}$$

$$\begin{aligned}
\Xi = \sum_{\mathbf{n}} \mathrm{Tr}\Big[&\frac{1}{\sqrt{2}}(\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}})[\overline{z}_{3,\mathbf{n}}, z_{3,\mathbf{n}}] + \frac{1}{\sqrt{2}}(\psi_{3,\mathbf{n}} + \lambda_{\mathbf{n}})[i\tilde{d}_{\mathbf{n}} - 2(\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i} z_{i,\mathbf{n}-\hat{\mathbf{e}}_i} - z_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}})]\\
&+ \xi_{3,\mathbf{n}}(\tilde{\overline{G}}_{\mathbf{n}} + 2\sqrt{2}\epsilon_{ij} z_{i,\mathbf{n}} z_{j,\mathbf{n}+\hat{\mathbf{e}}_i}) + \chi_{\mathbf{n}}(\tilde{G}_{\mathbf{n}} + 2\sqrt{2}\epsilon_{ij}\overline{z}_{i,\mathbf{n}+\hat{\mathbf{e}}_j}\overline{z}_{j,\mathbf{n}})\\
&+ \sqrt{2}\psi_{i,\mathbf{n}}(\overline{z}_{i,\mathbf{n}} z_{3,\mathbf{n}} - z_{3,\mathbf{n}+\hat{\mathbf{e}}_i}\overline{z}_{i,\mathbf{n}}) + \sqrt{2}\epsilon_{ij}\xi_{i,\mathbf{n}-\hat{\mathbf{e}}_j}(z_{j,\mathbf{n}-\hat{\mathbf{e}}_j} z_{3,\mathbf{n}} - z_{3,\mathbf{n}-\hat{\mathbf{e}}_j} z_{j,\mathbf{n}-\hat{\mathbf{e}}_j})\Big].
\end{aligned} \tag{2.15}$$

We will use this form to clarify the relationships.

2.2 The $\mathcal{N} = (2,2)$ Catterall model

Catterall's $\mathcal{N} = (2,2)$ supersymmetric lattice gauge theory [1] is based on the fact that the $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills theory can be regarded equivalently as a topological

[3] Be careful that there is the minus sign in eq. (2.10) which cannot be appear from the anti-commutation relation in the representation of super coordinates eq. (2.7). This difference comes from the fact that the left operation of supersymmetry group corresponds to the right motion in parameter space as described in the textbook written by Wess-Bagger [29].

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field theory [24]. The continuum action is thus expressed by an exact form by using a nilpotent supercharge Q

$$S = \beta Q \mathrm{Tr} \int d^2x \left(\frac{1}{4}\eta[\phi, \overline{\phi}] - 2i\chi_{12}F_{12} + \chi_{12}B_{12} + \psi_\mu D_\mu \overline{\phi} \right) \tag{2.16}$$

where ϕ, $\overline{\phi}$ are bosonic scalar fields, B_{12} is a bosonic anti-symmetric two tensor field, F_{12} is a field strength of vector gauge fields A_μ. η, ψ_μ, χ_{12} are fermion fields with one component spinor index. η is regarded as a scalar and ψ_μ are vectors and χ_{12} is an antisymmetric two-tensor under twisted rotational symmetry. Parameter β represent the inverse of the square of gauge coupling. Here, all fields are taken in the adjoint representation $C = \sum_a C^a T^a$ where T^a are *anti-hermitian generators* in the gauge group and C^a are real. The gauge symmetry is unitary $\mathrm{U}(M)$. D_μ is a covariant derivative with the adjoint representation using the anti-hermitian matrices A_μ. Indices μ run from 1 to 2 which represent the directions in two dimensional Euclidean space.

2.2.1 Catterall's lattice action

In constructing the lattice action, Catterall utilizes the Kahler-Dirac formalism and the lattice analogue of differential forms. He applied the criterion such that each scalars, vectors and antisymmetric two-tensors should be put on sites, links, and the plaquettes, respectively, on the lattice. Therefore, scalar fields ϕ, $\overline{\phi}$ and η are put on sites and vectors A_μ, ψ_μ reside on links and anti-symmetric two tensors χ_{12}, B_{12} reside on plaquettes. Then Catterall's action is given as[4]

$$\begin{aligned} S_L = -\beta Q \mathrm{Tr} \sum_x \Bigg(& \frac{1}{4}\eta^\dagger(x)[\phi(x), \bar{\phi}(x)] - i\chi_{12}^\dagger(x)\mathcal{F}_{12}(x) - i\chi_{12}(x)\mathcal{F}_{12}(x)^\dagger \\ & + \bigg(\frac{1}{2}\chi_{12}^\dagger(x)B_{12}(x) + \frac{1}{2}\chi_{12}(x)B_{12}^\dagger(x) \\ & + \frac{1}{2}\psi_\mu^\dagger(x)D_\mu^+\bar{\phi}(x) + \frac{1}{2}\psi_\mu(x)(D_\mu^+\bar{\phi}(x))^\dagger \bigg)\Bigg) \end{aligned} \tag{2.17}$$

where $U_{1,2}$ are bosonic link variables defined as $U_\mu = e^{A_\mu}$, and $\mathcal{F}_{12}$ are field strength of gauge fields defined as

$$\mathcal{F}_{12}(x) = D_1^+ U_2(x) = U_1(x)U_2(x+1) - U_2(x)U_1(x+2) \tag{2.18}$$

whose continuum limit is $F_{12}(x)$. D_μ^+ is covariant version of forward difference acting on a scalar field $f(x)$ and a vector field $f_\mu(x)$ as [28]

$$\begin{aligned} D_\mu^+ f(x) &= U_\mu(x)f(x+\mu) - f(x)U_\mu(x), \\ D_\mu^+ f_\nu(x) &= U_\mu(x)f_\nu(x+\mu) - f_\nu(x)U_\mu(x+\nu). \end{aligned} \tag{2.19}$$

[4]We change the notation of Catterall model a little bit. The difference from the original notation in his papers [1, 14] is as follows: In the action (2.17), we change the parameter β as $-\beta$. To take the continuum limit (2.16) consistent with the anti-hermitian condition $\eta^\dagger = -\eta$ imposed later on, this change is required. We also change the notation $\chi_{12}^\dagger\mathcal{F}_{12}$, $\chi_{12}\mathcal{F}_{12}^\dagger$ to $-i\chi_{12}^\dagger\mathcal{F}_{12}$, $-i\chi_{12}\mathcal{F}_{12}^\dagger$. By this change, the kinetic term of gauge field can be taken as positive definite $\mathcal{F}_{12}\mathcal{F}_{12}^\dagger$ after the integration of auxiliary fields $B_{12}B_{12}^\dagger$. If we do not change the notation, kinetic term of the gauge fields becomes $-\mathcal{F}_{12}\mathcal{F}_{12}^\dagger$. For the same reason, we also change the $2\chi_{12}\mathcal{F}_{12}$ to $-i2\chi_{12}\mathcal{F}_{12}$ in the target action (2.16).

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Note that, compared to the target theory (2.16), several new fields $\eta^\dagger$, $\phi^\dagger$, $\overline{\phi}^\dagger$, $\psi_\mu^\dagger$, $\chi^\dagger$ and $B_{12}^\dagger$, appear in his action. These conjugate fields transform as complex conjugate of original fields η, ϕ, $\overline{\phi}$, ψ_μ, χ and B_{12} under gauge transformation. Such conjugate fields are required to preserve the lattice gauge symmetry and naturally appear in the Kahler-Dirac formulation as described in section 3 of ref. [1].

His Q-transformation is defined by

$$\begin{aligned}
QU_\mu &= \psi_\mu & QU_\mu^\dagger &= \psi_\mu^\dagger, \\
Q\psi_\mu &= -D_\mu^+\phi, & Q\psi_\mu^\dagger &= -(D_\mu^+\phi)^\dagger, \\
Q\chi_{12} &= B_{12}, & Q\chi_{12}^\dagger &= B_{12}^\dagger, \\
QB_{12} &= [\phi,\chi_{12}]^{(12)}, & QB_{12}^\dagger &= \left([\phi,\chi_{12}]^{(12)}\right)^\dagger, \\
Q\overline{\phi} &= \eta, & Q\overline{\phi}^\dagger &= \eta^\dagger, \\
Q\eta &= [\phi,\overline{\phi}], & Q\eta^\dagger &= ([\phi,\overline{\phi}])^\dagger, \\
Q\phi &= 0,
\end{aligned} \tag{2.20}$$

where the superscript notation indicates a *shifted* commutator

$$[\phi,\chi_{\mu\nu}]^{(\mu\nu)} = \phi(x)\chi_{\mu\nu}(x) - \chi_{\mu\nu}(x)\phi(x+\mu+\nu). \tag{2.21}$$

Note that the Q-transformation laws satisfy following property

$$Q^2 = (\text{gauge transformation with the parameter } \phi). \tag{2.22}$$

2.2.2 Extra degrees of freedom in Catterall's theory

In the lattice action (2.17), there are extra degrees of freedom which the target theory does not have. Variables ϕ, $\overline{\phi}$, η, ψ_μ, χ, B_{12}, A_μ on the lattice are defined with general complex matrices $C = \sum_a (C^a + iD^a)T^a$, where C^a, D^a are real, while the variables in the target theory are defined with anti-hermitian matrices $C = \sum C^a T^a$. This is because the vector and tensor fields reside on the links and plaquettes, which are bi-fundamental representation under the lattice gauge group. The gauge transformation laws of generic vector fields $f_\mu(x)$ and anti-symmetric two tensors $f_{\mu\nu}(x)$ are assumed to be

$$\begin{aligned}
f_\mu(x) &\to V(x)f_\mu(x)V(x+\mu)^\dagger, \\
f_{\mu\nu}(x) &\to V(x)f_{\mu\nu}(x)V(x+\mu+\nu)^\dagger,
\end{aligned} \tag{2.23}$$

where $V(x)$, $V(x+\mu)$ and $V(x+\mu+\nu)$ are independent unitary matrices. Anti-hermiticity of the bi-fundamental variables cannot be maintained under the gauge transformation since the following equality is *not* always satisfied

$$\begin{aligned}
-(V(x)f_\mu(x)V(x+\mu)^\dagger)^\dagger &\equiv V(x+\mu)f_\mu(x)V(x)^\dagger = V(x)f_\mu(x)V(x+\mu)^\dagger, \\
-(V(x)f_{\mu\nu}(x)V(x+\mu+\nu)^\dagger)^\dagger &\equiv V(x+\mu+\nu)f_{\mu\nu}(x)V(x)^\dagger = V(x)f_{\mu\nu}(x)V(x+\mu+\nu)^\dagger,
\end{aligned}$$

due to the independence of gauge matrices $V(x)$ and $V(x+\mu)$, $V(x+\mu+\nu)$. Then such link and plaquette fields must be complexified as $(C^a + iD^a)T^a$. Therefore the new conjugate

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fields $\eta^\dagger, \phi^\dagger, \overline{\phi}^\dagger, \psi_\mu^\dagger, \chi^\dagger, B_{12}^\dagger$ are *independent* of η, ϕ, $\overline{\phi}$, ψ_μ, χ, B_{12}.[5] Link gauge field U_μ are also complexified, which are not unitary matrices, namely,

$$U_\mu(x)U_\mu^\dagger(x) \neq 1. \tag{2.24}$$

They are defined as $U_\mu(x) = e^{A_\mu(x)}$ with *complexified* gauge fields $A_\mu(x)$ whose hermitian conjugate $A_\mu^\dagger(x)$ are not equal to $-A_\mu(x)$.

The continuum limit of the action (2.17) is *different* from eq. (2.16). The degrees of freedom in eq. (2.17) are described with the general complex matrices $\sum_a (C^a + iD^a)T^a$ while the target theory (2.16) is defined with anti-hermitian matrices $\sum_a C^a T^a$.

2.3 Correspondence between $\mathcal{N} = (4,4)$ CKKU lattice theory and Catterall's action

If we neglect one $\mathcal{Q}$-multiplet, $\psi_{3,\mathbf{n}} + \lambda_\mathbf{n}$ and $\tilde{d}_\mathbf{n}$, the CKKU's $\mathcal{N}$ =(4,4) lattice action (2.15) and the $\mathcal{Q}$-transformations (2.12) are same as the Catterall's action (2.17) and his Q-transformations (2.20). One can check the equivalence by identifying the fields as follows:

$$\begin{aligned}
U_1(x) &\Leftrightarrow \sqrt{2}z_{1,\mathbf{n}}, & \psi_1(x) &\Leftrightarrow \sqrt{2}\psi_{1,\mathbf{n}}, \\
U_1^\dagger(x) &\Leftrightarrow \sqrt{2}\overline{z}_{1,\mathbf{n}}, & \psi_1^\dagger(x) &\Leftrightarrow -\sqrt{2}\xi_{2,\mathbf{n}} \\
U_2(x) &\Leftrightarrow \sqrt{2}z_{2,\mathbf{n}}, & \psi_2(x) &\Leftrightarrow \sqrt{2}\psi_{2,\mathbf{n}} \\
U_2^\dagger(x) &\Leftrightarrow \sqrt{2}\overline{z}_{2,\mathbf{n}}, & \psi_2^\dagger(x) &\Leftrightarrow \sqrt{2}\xi_{1,\mathbf{n}}, \\
\bar{\phi}(x) &\Leftrightarrow \sqrt{2}z_{3,\mathbf{n}}, & \phi(x) &\Leftrightarrow \sqrt{2}\overline{z}_{3,\mathbf{n}}, \\
\chi_{12}(x) &\Leftrightarrow -i\sqrt{2}\chi_\mathbf{n}, & \chi_{12}^\dagger(x) &\Leftrightarrow i\sqrt{2}\xi_{3,\mathbf{n}}, \\
B_{12}(x) &\Leftrightarrow -i\sqrt{2}\tilde{\overline{G}}_\mathbf{n}, & B_{12}^\dagger(x) &\Leftrightarrow i\sqrt{2}\tilde{G}_\mathbf{n}, \\
\mathcal{F}_{12}(x) &\Leftrightarrow 2\mathcal{E}_\mathbf{n}, & \mathcal{F}_{12}^\dagger(x) &\Leftrightarrow 2\mathcal{E}_\mathbf{n}^\dagger, \\
\eta(x) &\Leftrightarrow \sqrt{2}(\psi_{3,\mathbf{n}} - \lambda_\mathbf{n}),
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
\mathcal{E}_\mathbf{n} &= z_{1,\mathbf{n}} z_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - z_{2,\mathbf{n}} z_{1,\mathbf{n}+\hat{\mathbf{e}}_2}\,, \\
\mathcal{E}_\mathbf{n}^\dagger &= \overline{z}_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\overline{z}_{1,\mathbf{n}} - \overline{z}_{1,\mathbf{n}+\hat{\mathbf{e}}_2}\overline{z}_{2,\mathbf{n}}.
\end{aligned} \tag{2.26}$$

Here, in eq. (2.25), the left hand sides of the symbol $\Leftrightarrow$ are the fields in Catterall's theory and the right hand sides are fields in CKKU's theory.[6] Due to the complexification of the link and plaquette fields in "$\mathcal{N} = (2,2)$" Catterall's model, link fields $U_\mu(x)$,$U_\mu^\dagger(x)$, *etc*

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[5]It is not necessary to complexify the site fields η, ϕ, $\overline{\phi}$. They can keep the anti-hermiticity under the gauge transformation since they are in the adjoint representation. Therefore $\eta^\dagger = -\eta$, $\overline{\phi}^\dagger = -\overline{\phi}$ and $\phi^\dagger = -\phi$ can be taken on the lattice. Not only such a condition but also the condition $\eta^\dagger = \eta$, $\overline{\phi}^\dagger = \overline{\phi}$ and $\phi^\dagger = -\phi$ can be taken.

[6]In eq. (2.25), we impose the condition

$$\overline{\phi}^\dagger = \overline{\phi},\ \eta^\dagger = \eta,\ \phi^\dagger = -\phi, \tag{2.27}$$

on the site fields.

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of the Catterall's model can be identified as the complex link fields $z_{i,\mathbf{n}}$,$\overline{z}_{i,\mathbf{n}}$, *etc* of the "$\mathcal{N}=(4,4)$" CKKU model in the above correspondence. Note that $\tilde{d}_{\mathbf{n}}$ is a partner of $\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}}$ under the $\mathcal{Q}$-transformation

$$\mathcal{Q}\tilde{d}_{\mathbf{n}}=i[\overline{z}_{3,\mathbf{n}},\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}}],\quad \mathcal{Q}(\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}})=-\sqrt{2}i\tilde{d}_{\mathbf{n}}, \tag{2.28}$$

as in eq. (2.12). Other fields, except for $\overline{z}_{3,\mathbf{n}}$ whose $\mathcal{Q}$-transformation is $\mathcal{Q}\overline{z}_{3,\mathbf{n}}=0$, do not appear in this transformation. Therefore the absence of the set $\tilde{d}_{\mathbf{n}}$ and $\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}}$ does not affect the $\mathcal{Q}$-transformation laws of other fields. Moreover, since the set $\tilde{d}_{\mathbf{n}}$ and $\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}}$ exists only in one term

$$\mathcal{Q}\sum_{\mathbf{n}}\mathrm{Tr}\,\frac{1}{\sqrt{2}}(\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}})[i\tilde{d}_{\mathbf{n}}-2(\overline{z}_{i,\mathbf{n}-\hat{\mathbf{e}}_i}z_{i,\mathbf{n}-\hat{\mathbf{e}}_i}-z_{i,\mathbf{n}}\overline{z}_{i,\mathbf{n}})] \tag{2.29}$$

among the terms of CKKU action (2.15), the action (2.15) keep the $\mathcal{Q}$-exact form and the $\mathcal{Q}$-symmetry under the truncation. As a side remark, correspondences among the symbol of lattice sites and the gauge coupling of both theories are $x\Leftrightarrow\mathbf{n}$, $-\beta\Leftrightarrow\frac{1}{2g^2}$.

3. Relationship between Catterall model and Sugino's model

As described in section 2.2.2, Catterall's $\mathcal{N}=(2,2)$ action has extra degrees of freedom which do not present in the target $\mathcal{N}=(2,2)$ theory. Therefore it is necessary to truncate the extra degrees of freedom to identify his model with an $\mathcal{N}=(2,2)$ lattice model which contains the correct number of degrees of freedom of the target theory. If this is performed in a naive way, breaking of the supersymmetry on the lattice would be resulted. There exists a way of truncation which does not break the supersymmetry on the lattice. Then, we find, after this truncation, that the Catterall's theory *becomes the $\mathcal{N}=(2,2)$ supersymmetric lattice gauge theory being similar to the Sugino model in ref. [2].*

3.1 $\mathcal{N}=(2,2)$ lattice model by Sugino

To explain the derivation of the $\mathcal{N}=(2,2)$ Sugino type model from the Catterall's $\mathcal{N}=(2,2)$ lattice action, we briefly explain the Sugino's original $\mathcal{N}=(2,2)$ lattice model in [2].

His lattice action is

$$\begin{aligned}S^{\mathrm{LAT}}_{\mathcal{N}=2}=Q\frac{1}{2g_0^2}\sum_x\mathrm{Tr}\Bigg[&\frac{1}{4}\eta(x)\,[\phi(x),\,\bar{\phi}(x)]-i\chi(x)\Phi(x)+\chi(x)H(x)\\&+i\sum_{\mu=1}^{d}\psi'_\mu(x)\left(\bar{\phi}(x)-U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)\Bigg],\end{aligned} \tag{3.1}$$

where

$$\Phi(x)=-i\left[U_{12}(x)-U_{21}(x)\right], \tag{3.2}$$

and $U_{\mu\nu}(x)$ are plaquette variables

$$U_{\mu\nu}(x)\equiv U_\mu(x)U_\nu(x+\hat{\mu})U_\mu(x+\hat{\nu})^\dagger U_\nu(x)^\dagger.\quad(\mu,\nu=1,2) \tag{3.3}$$

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The target action of this model is the $\mathcal{N}=(2,2)$ super Yang-Mills action. In this model, the gauge fields are promoted to the compact unitary variables

$$U_\mu(x) = e^{iaA_\mu(x)} \tag{3.4}$$

on the link $(x, x+\mu)$. 'a' stands for the lattice spacing, and $x \in \mathbb{Z}^2$. All other fields sit on sites. Note that he uses the dimensionless variable in his paper. The Q-transformations of this model are as follows

$$\begin{aligned}
QU_\mu(x) &= i\psi'_\mu(x)U_\mu(x), \\
Q\psi'_\mu(x) &= i\psi'_\mu(x)\psi'_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right), \\
Q\phi(x) &= 0, \\
Q\vec{\chi}(x) &= \vec{H}(x), \qquad Q\vec{H}(x) = [\phi(x),\, \vec{\chi}(x)], \\
Q\bar{\phi}(x) &= \eta(x), \qquad Q\eta(x) = [\phi(x),\, \bar{\phi}(x)].
\end{aligned} \tag{3.5}$$

These Q-transformations satisfy following property

$$Q^2 = (\text{infinitesimal gauge transformation with the parameter } \phi\ ). \tag{3.6}$$

The action (3.1) is invariant under the Q-transformation since the action (3.1) is written by the Q-transformation of gauge invariant quantity. After the Q-operation, the action (3.1) takes the form

$$\begin{aligned}
S^{\mathrm{LAT}}_{\mathcal{N}=2} = \frac{1}{2g_0^2}\sum_x \mathrm{Tr}\, \Bigg[& \frac{1}{4}[\phi(x),\, \bar{\phi}(x)]^2 + H(x)H(x) - iH(x)\Phi(x) \\
& + \sum_{\mu=1}^{d}\left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) \\
& - \frac{1}{4}\eta(x)[\phi(x),\, \eta(x)] - \chi(x)[\phi(x),\, \chi(x)] \\
& - \sum_{\mu=1}^{d}\psi'_\mu(x)\psi'_\mu(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) \\
& + i\chi(x)Q\Phi(x) - i\sum_{\mu=1}^{d}\psi'_\mu(x)\left(\eta(x) - U_\mu(x)\eta(x+\hat{\mu})U_\mu(x)^\dagger\right)\Bigg].
\end{aligned}$$

3.2 Derivation of the $\mathcal{N}=(2,2)$ Sugino type model by a truncation of extra degrees of freedom in the Catterall's model

In this subsection, we show that the Catterall's $\mathcal{N}=(2,2)$ lattice model becomes $\mathcal{N}=(2,2)$ lattice model of the Sugino type if we truncate extra degrees of freedom in the Catterall's model by a way keeping supersymmetry on the lattice.

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We start from the Catterall action

$$S_L = -\beta Q\mathrm{Tr}\sum_x \Bigg(\frac{1}{4}\eta^\dagger(x)[\phi(x),\bar{\phi}(x)] - i\chi_{12}^\dagger(x)\mathcal{F}_{12}(x) - i\chi_{12}(x)\mathcal{F}_{12}(x)^\dagger$$
$$+\Bigg(\frac{1}{2}\chi_{12}^\dagger(x)B_{12}(x) + \frac{1}{2}\chi_{12}(x)B_{12}^\dagger(x)$$
$$+\frac{1}{2}\psi_\mu^\dagger(x)D_\mu^+\bar{\phi}(x) + \frac{1}{2}\psi_\mu(x)(D_\mu^+\bar{\phi}(x))^\dagger\Bigg)\Bigg), \tag{3.7}$$

and the Q-transformation laws (2.20)

$$\begin{aligned}
QU_\mu &= \psi_\mu & QU_\mu^\dagger &= \psi_\mu^\dagger, \\
Q\psi_\mu &= -D_\mu^+\phi, & Q\psi_\mu^\dagger &= -(D_\mu^+\phi)^\dagger, \\
Q\chi_{12} &= B_{12}, & Q\chi_{12}^\dagger &= B_{12}^\dagger, \\
QB_{12} &= [\phi,\chi_{12}]^{(12)}, & QB_{12}^\dagger &= \left([\phi,\chi_{12}]^{(12)}\right)^\dagger, \\
Q\overline{\phi} &= \eta, & Q\overline{\phi}^\dagger &= \eta^\dagger, \\
Q\eta &= [\phi,\overline{\phi}], & Q\eta^\dagger &= ([\phi,\overline{\phi}])^\dagger, \\
Q\phi &= 0.
\end{aligned} \tag{3.8}$$

If we can possess the following property of Q-transformation (2.22);

$$Q^2 = (\text{gauge transformation with parameter } \phi) \tag{3.9}$$

even after truncation, we can keep supersymmetry under the truncation.

To perform such truncation, we take the constraint $U_\mu(x)U_\mu^\dagger(x) = 1$, namely $A_\mu^\dagger(x) = -A_\mu(x)$ at first. Since $U_\mu(x)U_\mu^\dagger(x) = 1$ is not dynamical quantity, we obtain following conditions

$$Q(U_\mu(x)U_\mu^\dagger(x)) = 0, \quad QU_\mu(x) = \psi_\mu(x), \quad QU_\mu^\dagger(x) = \psi_\mu^\dagger(x). \tag{3.10}$$

By this condition, $\psi_\mu^\dagger(x)$ is described with $\psi_\mu(x)$ as

$$\psi_\mu^\dagger(x) = -U_\mu^\dagger(x)\psi_\mu(x)U_\mu^\dagger(x), \tag{3.11}$$

and $\psi_\mu^\dagger(x)$ is no longer independent of $\psi_\mu(x)$. Then if we define a site fermion fields $\psi'_\mu(x)$ as

$$\psi'_\mu(x) = \psi_\mu(x)U_\mu^\dagger(x), \tag{3.12}$$

two fermions $\psi_\mu(x)$ and $\psi_\mu^\dagger(x)$ are described only by one fermion variable $\psi'_\mu(x)$ as

$$\psi_\mu(x) = \psi'_\mu(x)U_\mu(x), \quad \psi_\mu^\dagger(x) = -U_\mu(x)^\dagger\psi'_\mu(x). \tag{3.13}$$

This $\psi'_\mu(x)$ becomes naturally *anti-hermitian* since

$$(\psi'_\mu(x))^\dagger = (\psi_\mu(x)U_\mu^\dagger(x))^\dagger = U_\mu(x)\psi_\mu^\dagger(x) = -\psi_\mu(x)U_\mu^\dagger(x) = -\psi'_\mu(x). \tag{3.14}$$

This anti-hermitian property can be kept under the gauge symmetry since the site fields are adjoint representation. Then we take $\psi'_\mu(x)$ as a fundamental fermionic variable rather than $\psi_\mu(x)$. From this expression and the Q-transformation of ψ_μ; $Q\psi_\mu(x) = \phi(x)U_\mu(x) - U_\mu(x)\phi(x+\mu)$, the Q-transformation law of $\psi'_\mu(x)$ is obtained naturally as

$$\begin{aligned} Q\psi'_\mu(x) &= (Q\psi_\mu(x)U^\dagger_\mu(x)) \\ &= (Q\psi_\mu(x))U^\dagger_\mu(x) - \psi_\mu(x)(QU^\dagger_\mu(x)) \\ &= \psi'_\mu(x)\psi'_\mu(x) + (\phi(x) - U_\mu(x)\phi(x+\mu)U^\dagger_\mu(x)). \end{aligned} \tag{3.15}$$

These conditions $U_\mu(x)U^\dagger_\mu(x) = 1$ and the eqs. (3.10) -(3.15) give a way to truncate extra degrees of freedom in gauge fields $U_\mu(x)$,$U^\dagger_\mu(x)$ and their partners $\psi_\mu(x)$, $\psi^\dagger_\mu(x)$ without breaking of the supersymmetry on the lattice.

For $\phi(x)$, $\overline{\phi}(x)$, $\eta(x)$, we impose $\eta^\dagger(x) = -\eta(x)$, $\overline{\phi}^\dagger(x) = -\overline{\phi}(x)$ and $\phi^\dagger(x) = -\phi(x)$ to remove the extra degrees of freedom. This condition can be kept under the gauge transformation since they are in adjoint representation. Since each $\overline{\phi}$ and $\overline{\phi}^\dagger$ compose the Q-multiplets with η and $\eta^\dagger$ respectively, this condition does not break the supersymmetry on the lattice.

To truncate extra degrees of freedom in χ_{12}, $\chi^\dagger_{12}$, B_{12} and $B^\dagger_{12}$ without breaking of the supersymmetry, we impose following constraints

$$\begin{aligned} \chi_{12}(x) &= \chi(x)U_1(x)U_2(x+1), \\ \chi^\dagger_{12}(x) &= -U^\dagger_2(x+1)U^\dagger_1(x)\chi(x), \\ B_{12}(x) &= H(x)U_1(x)U_2(x+1), \\ B^\dagger_{12}(x) &= -U^\dagger_2(x+1)U^\dagger_1(x)H(x). \end{aligned} \tag{3.16}$$

Here $\chi(x)$ and $H(x)$ are anti-hermitian site fields. $\chi(x)$ and $H(x)$ are obtained by absorbing the link gauge fields $U_\mu(x)$ to $\chi_{12}(x)$ and $B_{12}(x)$ as $\chi(x) = \chi_{12}(x)U_2(x+1)^\dagger U_1(x)^\dagger$ and $H(x) = B_{12}(x)U_2(x+1)^\dagger U_1(x)^\dagger$. By the above condition, $\chi^\dagger_{12}(x)$ and $B^\dagger_{12}(x)$ are no longer independent of $\chi_{12}(x)$ and $B_{12}(x)$, the degrees of freedom in these fields are reduced to two anti-hermite fields $\chi(x)$ and $H(x)$. By performing the Q-transformation on right hand sides and left hand sides of the above definitions eq. (3.16), one can immediately check that the Q-transformation on $\chi(x)$, $H(x)$;

$$Q\chi(x) = H(x) + \chi(x)\psi'_1(x) + \chi(x)U_1(x)\psi'_2(x+1)U^\dagger_1(x) \tag{3.17}$$

$$\begin{aligned} QH(x) &= \phi(x)\chi(x) - \chi(x)U_1(x)U_2(x+1)\phi(x+1+2)U^\dagger_2(x+1)U^\dagger_1(x) \\ &\quad -H(x)\psi'_1(x) - H(x)U_1(x)\psi'_2(x+1)U^\dagger_1(x) \end{aligned} \tag{3.18}$$

is obtained consistently. Note that Q^2 acts on these $\chi(x)$ and $H(x)$ as the infinitesimal gauge transformation with the parameter ϕ, namely

$$Q^2\chi(x) = [\phi(x), \chi(x)], \quad Q^2H(x) = [\phi(x), H(x)]. \tag{3.19}$$

By these conditions,

$$\frac{1}{2}\left(\chi^\dagger_{12}(x)B_{12}(x) + \chi_{12}(x)B^\dagger_{12}(x)\right) \tag{3.20}$$

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becomes

$$-\chi(x)H(x). \tag{3.21}$$

One can check it by substituting eq. (3.16) to the Catterall action (3.7). The term

$$-i\chi_{12}^{\dagger}(x)\mathcal{F}_{12}(x) - i\chi_{12}(x)\mathcal{F}_{12}^{\dagger}(x) \tag{3.22}$$

becomes

$$i\chi(x)\Phi(x) \tag{3.23}$$

where

$$\Phi(x) = U_1(x)U_2(x+\hat{1})U_1^{\dagger}(x+\hat{2})U_2^{\dagger}(x) - U_2(x)U_1(x+\hat{2})U_2^{\dagger}(x+\hat{1})U_1^{\dagger}(x). \tag{3.24}$$

Another term

$$+\frac{1}{2}\psi_{\mu}^{\dagger}(x)D_{\mu}^{+}\bar{\phi}(x) + \frac{1}{2}\psi_{\mu}(x)(D_{\mu}^{+}\bar{\phi}(x))^{\dagger} \tag{3.25}$$

becomes

$$-\psi'_{\mu}(x)(\bar{\phi}(x) - U_{\mu}\bar{\phi}(x+\mu)U_{\mu}^{\dagger}(x)). \tag{3.26}$$

It can also be checked by substitution of eq. (3.13) to the action (3.7). Therefore, Catterall's action (3.7) becomes

$$\begin{aligned} S_L = \beta Q \mathrm{Tr} \sum_x \Bigg(& \frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] + \chi(x)\Big(H(x) - i\Phi(x)\Big) \\ & -\psi'_{\mu}(x)(\bar{\phi}(x) - U_{\mu}\bar{\phi}(x+\mu)U_{\mu}^{\dagger}(x))\Bigg) \end{aligned} \tag{3.27}$$

in the truncation. The Q-transformation laws are

$$\begin{aligned} QU_{\mu}(x) &= \psi'_{\mu}(x)U_{\mu}(x), \\ Q\psi'_{\mu}(x) &= \psi'_{\mu}(x)\psi'_{\mu}(x) + (\phi(x) - U_{\mu}(x)\phi(x+\mu)U_{\mu}^{\dagger}(x)), \\ Q\chi(x) &= H(x) + \chi(x)\psi'_1(x) + \chi(x)U_1(x)\psi'_2(x+1)U_1^{\dagger}(x) \\ QH(x) &= \phi(x)\chi(x) - \chi(x)U_1(x)U_2(x+1)\phi(x+1+2)U_2^{\dagger}(x+1)U_1^{\dagger}(x) \\ &\quad -H(x)\psi'_1(x) - H(x)U_1(x)\psi'_2(x+1)U_1^{\dagger}(x) \\ Q\overline{\phi}(x) &= \eta(x), \\ Q\eta(x) &= [\phi(x),\overline{\phi}(x)], \\ Q\phi(x) &= 0. \end{aligned} \tag{3.28}$$

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The Q-transformation laws (3.8) become eq. (3.28) by the truncation. After the Q-operation, this action (3.27), becomes

$$\begin{aligned} S_L = \beta \sum_x \mathrm{Tr} \bigg(& \frac{1}{4}[\phi(x), \overline{\phi}(x)]^2 + H(x)(H(x) - i\Phi(x)) \\ & -\psi'_\mu(x)\psi'_\mu(x)(\bar{\phi}(x) + U_\mu \bar{\phi}(x+\mu)U_\mu^\dagger(x)) \\ & -(\phi(x) - U_\mu \phi(x+\mu)U_\mu^\dagger(x))(\bar{\phi}(x) - U_\mu \bar{\phi}(x+\mu)U_\mu^\dagger(x)) - \frac{1}{4}\eta(x)[\phi(x), \eta(x)] \\ & -\chi(x)\Big(\phi(x)\chi(x) - \chi(x)U_1(x)U_2(x+1)\phi(x+1+2)U_2^\dagger(x+1)U_1^\dagger(x)\Big) \\ & +i\chi(x)U_1(x)U_2(x+1)Q(U_2^\dagger(x+1)U_1^\dagger(x) - U_1^\dagger(x+2)U_2(x)) \\ & -i\chi(x)Q(U_1(x)U_2(x+1) - U_2(x)U_1(x+2))U_2^\dagger(x+1)U_1^\dagger(x) \\ & +\psi'_\mu(x)(\eta(x) - U_\mu \eta(x+\mu)U_\mu^\dagger(x))\bigg). \end{aligned} \tag{3.29}$$

This action eqs. (3.27), (3.29) has a correct continuum limit eq. (2.16) while the original Catterall action (3.7) does not have.

Note that the action (3.27), (3.29) is almost same as Sugino's action (3.1) and (3.7). Only the fermionic terms

$$-\chi(x)\Big(\phi(x)\chi(x) - \chi(x)U_1(x)U_2(x+1)\phi(x+1+2)U_2^\dagger(x+1)U_1^\dagger(x)\Big) \tag{3.30}$$

and

$$\begin{aligned} & +i\chi(x)U_1(x)U_2(x+1)Q(U_2^\dagger(x+1)U_1^\dagger(x) - U_1^\dagger(x+2)U_2(x)) \\ & -i\chi(x)Q(U_1(x)U_2(x+1) - U_2(x)U_1(x+2))U_2^\dagger(x+1)U_1^\dagger(x) \end{aligned} \tag{3.31}$$

are different from their corresponding terms $-\chi(x)[\phi(x), \chi(x)]$ and $i\chi(x)Q\Phi(x)$ in Sugino's original model (3.1), (3.7). After the integration over the auxiliary field $H(x)$, the gauge kinetic term

$$-\beta \sum_x \mathrm{Tr}\, \frac{1}{4}\Phi^2(x) = -\beta \sum_x \mathrm{Tr}\, \frac{1}{4}(U_{12}(x) - U_{21}(x))^2 \tag{3.32}$$

is obtained. This is same as the gauge kinetic term in Sugino's original model. Therefore, also the action (3.27), (3.29) has the vacuum degenerate problem which the original Sugino model encountered in ref. [2].[7]

Also the Q-transformation laws after the truncation (3.28) are almost same as the Q-transformation laws of the Sugino's model (3.5). Only the transformation laws of auxiliary field $H(x)$ and its partner $\chi(x)$ in (3.28) are different from the $Q\chi(x) = H(x)$ and $QH(x) = [\phi(x), \chi(x)]$ of Sugino's original model.

As a result, *Catterall's model becomes the theory which is almost same as the Sugino's theory by the truncation of extra degrees of freedom which does not break supersymmetry.*

[7]Sugino has proposed several treatments to solve this problem in refs. [2, 9].

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4. Relationship between Sugino's model and CKKU model

Due to the two relationships described in section 2 and 3, it is obvious that the model of the Sugino type can be derived by the truncation of degrees of freedom in the CKKU model. Due to the relationship between CKKU model and Catterall's model, the method to derive the Sugino type model from the Catterall's theory is applicable to derive the model of the Sugino type from CKKU model. Since the explanation of the derivation is mere repetition of the description in the subsection 3.2, we put off the explanation of the derivation in the appendix A.

In this section, we explain that the derivation discards the fluctuations along the flat-direction around the vacuum expectation value $\frac{1}{\sqrt{2}a}$ of scalar potential existing in the CKKU model.[8]

To explain it, we explain the deconstruction and the fluctuation in the CKKU model at first. Then we explain that the derivation truncates such fluctuations.

4.1 The deconstruction and the fluctuations along the moduli space in CKKU model

To realize the kinetic term in CKKU model, performing the "deconstruction" is required.(see also section 3.3 in ref. [3].) The deconstruction is the field redefinition of the bosonic link fields $z_{i,\mathbf{n}}$ expanding around the vacuum expectation value

$$\langle z_{i,\mathbf{n}} \rangle = \frac{1}{\sqrt{2}a} \mathbf{1}_M, \tag{4.1}$$

where the $\mathbf{1}_M$ is $M \times M$ unit matrix and the a is interpreted as lattice spacing.

To perform the expansions, there are two ways of representations; Cartesian decomposition and the polar decomposition. These two decomposition give the same continuum limit as Unsal proved in ref. [30]. CKKU adopts the Cartesian decomposition, eq. (3.16) in ref. [3], which represents the complex link variables by the sum of hermitian matrices and the antihermitian matrices. But, to perform the derivation of the Sugino type model, we have to adopt the *polar decomposition* [30−32].

In the polar decomposition, the bosonic link fields z_i,$\overline{z}_i$ $(i = 1, 2)$ are uniquely represented as a product of hermitian matrices $(\frac{1}{a} + s_{i,\mathbf{n}})$ $(i = 1, 2)$, which represent a radial direction and so have positive eigenvalues only, and unitary matrices $U_{i,\mathbf{n}}$

$$z_{i,\mathbf{n}} = \frac{1}{\sqrt{2}} \left(\frac{1}{a} \mathbf{1}_M + s_{i,\mathbf{n}} \right) U_{i,\mathbf{n}}, \quad \overline{z}_{i,\mathbf{n}} = \frac{1}{\sqrt{2}} U^\dagger_{i,\mathbf{n}} \left(\frac{1}{a} \mathbf{1}_M + s_{i,\mathbf{n}} \right), \tag{4.2}$$

where lattice spacing $\frac{1}{\sqrt{2}a}$ and the scalar fields $s_{i,\mathbf{n}}$ sit on sites and $U_{i,\mathbf{n}}$ are link fields written by the gauge fields $v_{i,\mathbf{n}}$ as $U_{i,\mathbf{n}} = e^{iav_{i,\mathbf{n}}}$. Comparing with the Cartesian decomposition in ref. [3], this representation of decomposition has advantage of the manifest gauge symmetry. This representation is required to keep the gauge symmetry under the truncation.

[8] In this section, we take into account the the flat-directions of moduli space while we neglect such effects in section 2; we showed the relationship between the CKKU model and the Catterall model without the consideration of such effects.

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Note that, in the CKKU model, the lattice spacing is dynamical quantity characterized as the vacuum expectation value $\frac{1}{\sqrt{2}a}$ of scalar potential. The scalar fields s_i are fluctuations around the $\frac{1}{\sqrt{2}a}$.

The CKKU action has noncompact moduli space consisting of all constant scalar fields satisfying $[s_1, s_2] = 0$. The integral of these modes are formally divergent, the expansion (4.2) is then poorly defined. (Even if we take the Cartesian decomposition taken in ref. [3], such instability of the vacuum occurs.) To suppress the large fluctuation along the flat directions, the original CKKU model introduced the moduli fixing mass term

$$\sum_{\mathbf{n}} \mathrm{Tr}\left[\left(z_{i,\mathbf{n}}\bar{z}_{i,\mathbf{n}} - \frac{1}{2a^2}\right)^2\right] = \sum_{\mathbf{n}} \mathrm{Tr}\,\frac{1}{4}\left[\left(\left(s_{i,\mathbf{n}} + \frac{1}{a}\right)^2 - \frac{1}{a^2}\right)^2\right]. \tag{4.3}$$

4.2 Truncation of the flat-direction by the derivation of the Sugino type model

When we derived the model of the Sugino type from the Catterall model, we imposed the condition that the link variables become unitary, namely,

$$U_\mu(x)U_\mu^\dagger(x) = 1. \tag{4.4}$$

Therefore, from the correspondence between the fields of CKKU model and the ones of Catterall model (2.25), complex link fields $z_{i,\mathbf{n}}$ in the CKKU model become "unitary" link variables to derive the model of the Sugino type. This means that dynamical degrees of freedom which correspond to radial directions of the links z_i, $\bar{z}_i$ are discarded in the derivation, namely,

$$z_{i,\mathbf{n}} = \frac{1}{\sqrt{2}a}U_{i,\mathbf{n}}, \quad \bar{z}_{i,\mathbf{n}} = \frac{1}{\sqrt{2}a}U^\dagger_{i,\mathbf{n}}, \tag{4.5}$$

where the vacuum expectation value $\frac{1}{\sqrt{2}a}$ cannot be removed since the link fields $z_i, \bar{z}_i$ have mass dimension 1.

Note that the derivation of the Sugino type model from the CKKU model discards the fluctuations s_i around the vacuum expectation value $\frac{1}{\sqrt{2}a}$. Then, also the large fluctuations of s_i along the flat-directions which cause the serious instability of the vacuum are removed under the derivation. Therefore we do not have to introduce the moduli fixing mass term in the derived Sugino type model. Moreover we can derive the model of the Sugino type from the CKKU model even if we introduce the moduli fixing mass term (4.3),

$$\sum_{\mathbf{n}} \mathrm{Tr}\left[\left(z_{i,\mathbf{n}}\bar{z}_{i,\mathbf{n}} - \frac{1}{2a^2}\right)^2\right] = \sum_{\mathbf{n}} \mathrm{Tr}\,\frac{1}{4}\left[\left(\left(s_{i,\mathbf{n}} + \frac{1}{a}\right)^2 - \frac{1}{a^2}\right)^2\right] \tag{4.6}$$

in the CKKU model. This is because the mass term naturally vanishes under the truncation $s_{i,\mathbf{n}} = 0$.

5. Corresponding truncation in the continuum theory

The derivation of the Sugino type model from the CKKU model (or Catterall model) can be interpreted as the lattice analogue of the derivation of the continuum $\mathcal{N} = (2,2)$ theory from the continuum $\mathcal{N} = (4,4)$ theory by the truncation of several Q-multiplets.

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We first consider the continuum $\mathcal{N} = (4,4)$ supersymmetric gauge theory action

$$S = \frac{1}{g_2^2} \int d^2x Q\Xi \tag{5.1}$$

where

$$\Xi = \mathrm{Tr}\left[\frac{1}{4}\eta[\phi,\overline{\phi}] + \chi^{\mathbb{R}}(H^{\mathbb{R}} - i\mathcal{E}^{\mathbb{R}}) + \chi_1(H_1 - i\mathcal{E}_1) + \chi_2(H_2 - i\mathcal{E}_2) + \frac{1}{2}\left\{\psi_\mu D_\mu\overline{\phi} + \psi_{s_i}[s_i,\overline{\phi}]\right\}\right],$$

and

$$\begin{aligned}
\mathcal{E}^{\mathbb{R}} &= -2(D_1 s_1 + D_2 s_2), \\
\mathcal{E}_1 &= 2(D_1 s_2 - D_2 s_1), \\
\mathcal{E}_2 &= 2(i[s_1, s_2] + F_{12}), \\
F_{12} &= -i[D_1, D_2].
\end{aligned}$$

Here the indices μ, i run from 1 to 2, and the repeated indices are summed. s_i, ϕ, $\overline{\phi}$ are bosonic scalar fields and $H^{\mathbb{R}}$, H_i are auxiliary fields and v_μ are gauge fields. The others ψ_{s_i}, ψ_μ, $\chi^{\mathbb{R}}$, χ_i are fermionic fields. The all fields are in adjoint representation of the gauge group. D_μ is the covariant derivative. Q-transformation laws are

$$\begin{aligned}
Qs_i &= (\psi_{s_i}), & Q\psi_{s_i} &= [\overline{\phi}, s_i], \\
Q\overline{\phi} &= \eta, & Q\eta &= [\phi, \overline{\phi}], \\
Qv_\mu &= \psi_\mu, & Q\psi_\mu &= iD_\mu\phi, \\
Q\chi^{\mathbb{R}} &= H^{\mathbb{R}}, & QH^{\mathbb{R}} &= [\phi, \chi^{\mathbb{R}}], \\
Q\chi_i &= H_i, & QH_i &= [\phi, \chi_i]\ (i = 1, 2), \\
Q\phi &= 0.
\end{aligned} \tag{5.2}$$

In each Q-transformation law of each Q-multiplet, only the components of the multiplet and ϕ, whose transformation is $Q\phi = 0$, appear. Also note that the twice operation of Q generates the infinitesimal gauge transformation with the parameter ϕ. The action is invariant under the Q-operation since it is written as Q-transformation of the gauge invariant quantity.

To derive the $\mathcal{N} = (2,2)$ supersymmetric theory from the $\mathcal{N} = (4,4)$ theory, we discard following Q-multiplets;

1. $\chi^{\mathbb{R}}$ and $H^{\mathbb{R}}$, which are contained only in the term $\chi^R(H^{\mathbb{R}} - i\mathcal{E}^{\mathbb{R}})$ among the terms in Ξ of eq. (5.1)

2. χ_1 and H_1 contained only in the term $\chi_1(H_1 - i\mathcal{E}_1)$

3. s_i and ψ_{s_i} contained only in $\frac{1}{2}\psi_{s_i}[s_i, \phi]$ and $2\chi_2[s_1, s_2]$.

If we substitute the condition

$$s_i = \psi_{s_i} = \chi^{\mathbb{R}} = H^{\mathbb{R}} = H_1 = \chi_1 = 0, \tag{5.3}$$

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the action (5.1) reduces to

$$S_{(2,2)} = \frac{1}{g_2^2}\int d^2x Q\Xi' \tag{5.4}$$
$$\Xi' = \mathrm{Tr}\left[\frac{1}{4}\eta[\phi,\overline{\phi}] + \chi(H - i\mathcal{E}) + \frac{1}{2}\left\{\psi_\mu D_\mu \overline{\phi}\right\}\right]$$

where

$$\mathcal{E} = 2(F_{12}),$$
$$F_{12} = -i[D_1, D_2].$$

Ξ in eq. (5.1) reduces to the above gauge invariant quantity Ξ' under the truncation. Note that the absence of the three Q-multiplets s_i and ψ_{s_i}, *etc* does not change the Q-transformations of other fields. This is because only the remaining fields η, $\overline{\phi}$, *etc*, which survive under the truncation, appear in the Q-transformation laws of the remaining fields. Moreover the condition (5.3) is kept under the Q-transformation. Also note this action (5.4) is also written as the Q-operation on the gauge invariant quantity Ξ'. Therefore the action (5.4) keeps the Q-symmetry. This action (5.4) is equivalent to the continuum $\mathcal{N} = (2,2)$ supersymmetric gauge theory. Finally, we obtain the $\mathcal{N} = (2,2)$ theory by the truncation of degrees of freedom in the $\mathcal{N} = (4,4)$ theory.

The derivation of the Sugino type model from the CKKU model (or Catterall model) is the lattice analogue of the derivation of this continuum $\mathcal{N} = (2,2)$ theory (5.4) from $\mathcal{N} = (4,4)$ theory.

6. Conclusion

In this paper, we clarified the relationship between several, seemingly quite different, supersymmetric lattice models preserving supersymmetry on the lattice. First we showed that Catterall's model can be embedded in CKKU's model as a sub-sector. Also we clarified that a model of the Sugino type naturally appears when we truncate the degrees of freedom in Catterall's model in a way which does not break the supersymmetry on the lattice. We also show that the $\mathcal{N} = (4,4)$ CKKU model can give the Sugino type model if we truncate the fluctuations around the vacuum expectation value $\frac{1}{\sqrt{2}a}$ and other degrees of freedom.

These relationships would indicate an underlying essential structure which any lattice formulations preserving partial supersymmetry possess. Further understanding of this structure would be very useful to develop lattice formulations of supersymmetric gauge theory.

Since the Catterall's lattice model and the model of Sugino type can be built also from the CKKU lattice model which is constructed from super matrix model, we would be able to utilize the super matrix model analysis for these lattice formulations. There is a possibility that also Catterall's model and the Sugino type model could be described using the matrix model analysis.

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A. Derivation of the $\mathcal{N} = (2,2)$ Sugino type model from CKKU model

We will explicitly show the derivation of the Sugino type model from the $\mathcal{N} = (4,4)$ CKKU lattice model. Here we utilize the technology to derive the $\mathcal{N} = (2,2)$ lattice theory from $\mathcal{N} = (4,4)$ lattice theory proposed in ref. [32].

At first, we truncate some scalars and auxiliary fields by imposing $\tilde{d}_{\mathbf{n}} = s_i = \psi_{3,\mathbf{n}} + \lambda_{\mathbf{n}} = 0$. After this truncation, the expansion eq. (4.2) of the bosonic link fields $z_{i,\mathbf{n}}$, $\bar{z}_{i,\mathbf{n}}$ become

$$z_{i,\mathbf{n}} = \frac{1}{\sqrt{2}a} U_{i,\mathbf{n}}, \quad \bar{z}_{i,\mathbf{n}} = \frac{1}{\sqrt{2}a} U^{\dagger}_{i,\mathbf{n}}. \tag{A.1}$$

Note that $\bar{z}_{i,\mathbf{n}}$ are no longer independent of $z_{i,\mathbf{n}}$ due to the absence of the scalar fields $s_{i,\mathbf{n}}$. Since only the scalar fields s_i can give the dynamical fluctuations and the radiative corrections of lattice spacing, the lattice spacing $\frac{1}{\sqrt{2}a}$ is no longer dynamical quantity. The product of these two link fields $z_{i,\mathbf{n}}$ and the $\bar{z}_{i,\mathbf{n}}$ combine the *non-dynamical* lattice spacing as $z_{i,\mathbf{n}} \bar{z}_{i,\mathbf{n}} = \frac{1}{2a^2}$. Therefore, we can take a condition that the $\mathcal{Q}$-transformation of the product $z_{i,\mathbf{n}} \bar{z}_{i,\mathbf{n}} = \frac{1}{2a^2}$ vanishes. Thus, from eq. (A.1), we immediately obtain

$$\begin{aligned} \mathcal{Q} z_{i,\mathbf{n}} &= \frac{1}{\sqrt{2}a} \mathcal{Q} U_{i,\mathbf{n}} = \psi_{i,\mathbf{n}}, \qquad \mathcal{Q} \bar{z}_{i,\mathbf{n}} = \frac{1}{\sqrt{2}a} \mathcal{Q} U^{\dagger}_{i,\mathbf{n}} = -\epsilon^{ij} \xi_{j,\mathbf{n}}, \\ \mathcal{Q}(z_{i,\mathbf{n}} \bar{z}_{i,\mathbf{n}}) &= \mathcal{Q} \frac{1}{2a^2} = 0. \end{aligned} \tag{A.2}$$

From eqs. (A.1), (A.2), we obtain the constraints between fermions ψ_i and $\epsilon_{ij}\xi_j$

$$-\epsilon_{ij} \xi_{j,\mathbf{n}} = -U^{\dagger}_{x,\mathbf{n}} \psi_{i,\mathbf{n}} U^{\dagger}_{x,\mathbf{n}}. \tag{A.3}$$

By this definition, half of degrees of freedom in complex fermion fields $\psi_{i,\mathbf{n}}$ and $\xi_{i,\mathbf{n}}$ are discarded. $\xi_{i,\mathbf{n}}$ are no longer independent of $\psi_{i,\mathbf{n}}$. Due to the relationships (A.3), we can represent the above link fermions $\psi_{i,\mathbf{n}}$, $\xi_{i,\mathbf{n}}$ by absorbing the link variables as

$$\psi_{i,\mathbf{n}} = i \psi^i_{\mathbf{n}} U_{i,\mathbf{n}}, \tag{A.4}$$

$$-\epsilon_{ij} \xi_{j,\mathbf{n}} = -i U^{\dagger}_{i,\mathbf{n}} \psi^i_{\mathbf{n}}, \tag{A.5}$$

where $\psi^i_{\mathbf{n}}$ are site fermions in the adjoint representation. The $\mathcal{Q}$-transformations of the site fermions $\psi^i_{\mathbf{n}}$ are naturally obtained as

$$\mathcal{Q} \psi^i_{\mathbf{n}} = \sqrt{2}a\, \psi^i_{\mathbf{n}} \psi^i_{\mathbf{n}} - i \frac{1}{a} \left(\bar{z}_{3,\mathbf{n}} - U_{i,\mathbf{n}} \bar{z}_{3,\mathbf{n}+\mathbf{i}} U^{\dagger}_{i,\mathbf{n}} \right). \tag{A.6}$$

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We can also discard the half of degrees of freedom in the fermion fields $\chi_{\mathbf{n}}$ and $\xi_{3,\mathbf{n}}$ by imposing the condition

$$\chi_{\mathbf{n}} = -\chi'_{\mathbf{n}} U_{1,\mathbf{n}} U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}, \tag{A.7}$$

$$\xi_{3,\mathbf{n}} = -U^{\dagger}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} \chi'_{\mathbf{n}}, \tag{A.8}$$

where $\chi'_{\mathbf{n}}$ is a site fermion. By this condition, $\chi_{\mathbf{n}}$ is no longer independent of $\xi_{3,\mathbf{n}}$. The truncation of the degrees of freedom in the bosonic auxiliary fields $\tilde{\tilde{G}}_{\mathbf{n}}, \tilde{G}_{\mathbf{n}}$ are also performed by absorbing the link fields as,

$$\begin{aligned}
\tilde{G}_{\mathbf{n}} = \mathcal{Q}\xi_{3,\mathbf{n}} &\equiv -U^{\dagger}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} H_{\mathbf{n}} \\
&= -(\sqrt{2}a\xi_{1,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} \chi'_{\mathbf{n}} - \sqrt{2}a U^{\dagger}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} \xi_{2,\mathbf{n}} \chi'_{\mathbf{n}} + U^{\dagger}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} \mathcal{Q}\chi'_{\mathbf{n}}), \\
\tilde{\tilde{G}}_{\mathbf{n}} = \mathcal{Q}\chi_{\mathbf{n}} &\equiv -H_{\mathbf{n}} U_{1,\mathbf{n}} U_{2,\mathbf{n}+\hat{\mathbf{e}}_1} \\
&= -(\mathcal{Q}\chi'_{\mathbf{n}} U_{1,\mathbf{n}} U_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \sqrt{2}a\chi'_{\mathbf{n}} \psi_{1,\mathbf{n}} U_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - \sqrt{2}a\chi'_{\mathbf{n}} U_{1,\mathbf{n}} \psi_{2,\mathbf{n}+\hat{\mathbf{e}}_1}),
\end{aligned} \tag{A.9}$$

where $H_{\mathbf{n}}$ is a bosonic site field. The $\mathcal{Q}$-transformation laws of $\chi'_{\mathbf{n}}$ and $H_{\mathbf{n}}$ are

$$\begin{aligned}
\mathcal{Q}\chi'_{\mathbf{n}} &= H_{\mathbf{n}} + i\sqrt{2}a\psi^1_{\mathbf{n}} \chi'_{\mathbf{n}} + i\sqrt{2}a U_{1,\mathbf{n}} \psi^2_{\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} \chi'_{\mathbf{n}}, \\
\mathcal{Q}H_{\mathbf{n}} &= \sqrt{2}\Bigg(-(\chi'_{\mathbf{n}} \bar{z}_{3,\mathbf{n}} - U_{1,\mathbf{n}} U_{2,\mathbf{n}+\hat{\mathbf{e}}_1} \bar{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2} U^{\dagger}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} \chi'_{\mathbf{n}}) \\
&\qquad + iaU_{1,\mathbf{n}} \psi^2_{\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} H_{\mathbf{n}} + ia\psi^1_{\mathbf{n}} H_{\mathbf{n}} \Bigg).
\end{aligned} \tag{A.10}$$

The above conditions eqs. (A.2)–(A.10) in $\mathcal{N} = (4,4)$ CKKU lattice theory are almost same as the truncation conditions eqs. (3.10)–(3.18) which derive the model of the Sugino type from Catterall's model in the subsection 3.2. Then, the property

$$\mathcal{Q}^2 = (\text{infinitesimal gauge transformation with parameter } \bar{z}_3) \tag{A.11}$$

is kept even after the truncations. Therefore, the $\mathcal{N} = (4,4)$ CKKU lattice action can be truncated to $\mathcal{N} = (2,2)$ lattice action with a preserved supercharge $\mathcal{Q}$. The $\mathcal{N} = (2,2)$ lattice action is described as follows,

$$S = \frac{1}{2g^2} \mathcal{Q}\,\Xi' \tag{A.12}$$

where

$$\Xi' = \sum_{\mathbf{n}} \mathrm{Tr}\Bigg[\frac{1}{\sqrt{2}} (\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}) [\bar{z}_{3,\mathbf{n}}, z_{3,\mathbf{n}}] \tag{A.13}$$

$$\qquad + 2\chi'_{\mathbf{n}} H_{\mathbf{n}} - i\frac{\sqrt{2}}{a^2} \chi'_{\mathbf{n}} (\Phi_{\mathbf{n}}) + \frac{2i}{a} \psi^i_{\mathbf{n}} (z_{3,\mathbf{n}} - U_{i,\mathbf{n}} z_{3,\mathbf{n}+\hat{\mathbf{e}}_i} U^{\dagger}_{i,\mathbf{n}}) \Bigg], \tag{A.14}$$

$$\Phi_{\mathbf{n}} = -i\left(U_{1,\mathbf{n}} U_{2,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}+\hat{\mathbf{e}}_2} U^{\dagger}_{2,\mathbf{n}} - U_{2,\mathbf{n}} U_{1,\mathbf{n}+\hat{\mathbf{e}}_2} U^{\dagger}_{2,\mathbf{n}+\hat{\mathbf{e}}_1} U^{\dagger}_{1,\mathbf{n}} \right). \tag{A.15}$$

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The $\mathcal{Q}$-transformations of the fields in eq. (A.15) are summarized as

$$\begin{aligned}
&\mathcal{Q}U_{i,\mathbf{n}} = i\psi^i_{\mathbf{n}}U_{i,\mathbf{n}},\\
&\mathcal{Q}\psi^i_{\mathbf{n}} = \sqrt{2}a\psi^i_{\mathbf{n}}\psi^i_{\mathbf{n}} - i\frac{1}{a}\left(\overline{z}_{3,\mathbf{n}} - U_{i,\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_i}U^\dagger_{i,\mathbf{n}}\right),\\
&\mathcal{Q}\overline{z}_{3,\mathbf{n}} = 0,\\
&\mathcal{Q}\chi'_{\mathbf{n}} = H_{\mathbf{n}} + i\sqrt{2}a\psi^1_{\mathbf{n}}\chi'_{\mathbf{n}} + i\sqrt{2}aU_{1,\mathbf{n}}\psi^2_{\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}}\chi'_{\mathbf{n}},\\
&\mathcal{Q}H_{\mathbf{n}} = \sqrt{2}\Big(- (\chi'_{\mathbf{n}}\overline{z}_{3,\mathbf{n}} - U_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}}\chi'_{\mathbf{n}})\\
&\qquad\qquad +iaU_{1,\mathbf{n}}\psi^2_{\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}}H_{\mathbf{n}} + ia\psi^1_{\mathbf{n}}H_{\mathbf{n}}\Big),\\
&\mathcal{Q}z_{3,\mathbf{n}} = \psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}, \quad \mathcal{Q}(\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}) = \sqrt{2}[\overline{z}_{3,\mathbf{n}},\, z_{3,\mathbf{n}}].
\end{aligned} \tag{A.16}$$

After the $\mathcal{Q}$-operation, the action becomes

$$\begin{aligned}
S = \frac{1}{2g^2}\sum_{\mathbf{n}}\mathrm{Tr}\Bigg[&[\overline{z}_{3,\mathbf{n}},\, z_{3,\mathbf{n}}]^2 + 2H_{\mathbf{n}}H_{\mathbf{n}} - i\frac{\sqrt{2}}{a^2}H_{\mathbf{n}}\Phi_{\mathbf{n}}\\
&+\sum_{i=1}^{2}\frac{2}{a^2}\left(\overline{z}_{3,\mathbf{n}} - U_{i,\mathbf{n}}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_i}U^\dagger_{i,\mathbf{n}}\right)\left(z_{3,\mathbf{n}} - U_{i,\mathbf{n}}z_{3,\mathbf{n}+\hat{\mathbf{e}}_i}U^\dagger_{i,\mathbf{n}}\right)\\
&-\frac{1}{\sqrt{2}}(\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}})[\overline{z}_{3,\mathbf{n}},\, (\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}})]\\
&-2\sqrt{2}\chi'_{\mathbf{n}}(\overline{z}_{3,\mathbf{n}}\chi'_{\mathbf{n}} - \chi'_{\mathbf{n}}U^\dagger_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\overline{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}})\\
&-\sum_{\mu=1}^{2}2\sqrt{2}\psi^i_{\mathbf{n}}\psi^i_{\mathbf{n}}\left(z_{3,\mathbf{n}} + U_{i,\mathbf{n}}z_{3,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{i,\mathbf{n}}\right)\\
&+\frac{\sqrt{2}}{a^2}\chi'_{\mathbf{n}}U_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\mathcal{Q}(U^\dagger_{1,\mathbf{n}+\hat{\mathbf{e}}_2}U^\dagger_{2,\mathbf{n}} - U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}})\\
&+\frac{\sqrt{2}}{a^2}\chi'_{\mathbf{n}}\mathcal{Q}(U_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1} - U_{2,\mathbf{n}}U_{1,\mathbf{n}+\hat{\mathbf{e}}_2})U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}}\\
&-i\sum_{i=1}^{2}\frac{2}{a}\psi^i_{\mathbf{n}}\left((\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}) - U_{i,\mathbf{n}}(\psi_{3,\mathbf{n}+\hat{\mathbf{e}}_i} - \lambda_{\mathbf{n}+\hat{\mathbf{e}}_i})U^\dagger_{i,\mathbf{n}}\right)\Bigg].
\end{aligned} \tag{A.17}$$

One can confirm that the $\mathcal{Q}$-transformation (A.16) and the action (A.15), (A.17) are almost same as Q-transformation in the Sugino's model (3.5) and his action (3.1), (3.7), by following identifications,

$$\begin{aligned}
\eta(x) &\Leftrightarrow \sqrt{2}a^{3/2}(\psi_{3,\mathbf{n}} - \lambda_{\mathbf{n}}), & \overline{\phi}(x) &\Leftrightarrow \sqrt{2}a\, z_{3,\mathbf{n}},\\
\chi(x) &\Leftrightarrow \sqrt{2}a^{3/2}\chi'_{\mathbf{n}}, & H(x) &\Leftrightarrow \sqrt{2}a^2H_{\mathbf{n}},\\
U_\mu(x) &\Leftrightarrow U_{i,\mathbf{n}}, & \psi'_\mu(x) &\Leftrightarrow \sqrt{2}a^{3/2}\psi^i_{\mathbf{n}},\\
\phi(x) &\Leftrightarrow \sqrt{2}a\,\overline{z}_{3,\mathbf{n}}, & &\\
\mu &\Leftrightarrow i, & Q &\Leftrightarrow a^{1/2}\mathcal{Q},\\
\frac{1}{2g_0^2} &\Leftrightarrow \frac{1}{2g^2}a^{-4}. & &
\end{aligned} \tag{A.18}$$

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Only the several fermionic terms in eq. (A.17)

$$+\frac{\sqrt{2}}{a^2}\chi'_{\mathbf{n}}U_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\mathcal{Q}(U^\dagger_{1,\mathbf{n}+\hat{\mathbf{e}}_2}U^\dagger_{2,\mathbf{n}}-U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}})$$
$$+\frac{\sqrt{2}}{a^2}\chi'_{\mathbf{n}}\mathcal{Q}(U_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}-U_{2,\mathbf{n}}U_{1,\mathbf{n}+\hat{\mathbf{e}}_2})U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}} \tag{A.19}$$

and

$$-2\sqrt{2}\chi'_{\mathbf{n}}(\bar{z}_{3,\mathbf{n}}\chi'_{\mathbf{n}}-\chi'_{\mathbf{n}}U^\dagger_{1,\mathbf{n}}U_{2,\mathbf{n}+\hat{\mathbf{e}}_1}\bar{z}_{3,\mathbf{n}+\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2}U^\dagger_{2,\mathbf{n}+\hat{\mathbf{e}}_1}U^\dagger_{1,\mathbf{n}}) \tag{A.20}$$

are different from their corresponding terms $\eta(x)[\phi(x),\eta(x)]$ and $-i\chi(x)Q\Phi(x)$ in the original Sugino model (3.7). In the $\mathcal{Q}$-transformation laws eq. (A.16), only the transformation laws of auxiliary fields and their partner $\chi'_{\mathbf{n}}$ are different from ones of the Sugino's model (3.5). Then the $\mathcal{N}=(2,2)$ lattice gauge theory of the Sugino type is derived from the $\mathcal{N}=(4,4)$ CKKU lattice theory by the suitable truncation of fields.

Although we have derived from the CKKU model without moduli fixing mass term, we can derive the same model even if we introduce the moduli fixing mass term

$$\sum_{\mathbf{n}}\mathrm{Tr}\left[\left(z_{i,\mathbf{n}}\bar{z}_{i,\mathbf{n}}-\frac{1}{2a^2}\right)^2\right]=\sum_{\mathbf{n}}\mathrm{Tr}\,\frac{1}{4}\left[\left(\left(s_{i,\mathbf{n}}+\frac{1}{a}\right)^2-\frac{1}{a^2}\right)^2\right] \tag{A.21}$$

in the CKKU model. This is because such mass term naturally vanishes under the truncation $s_{i,\mathbf{n}}=0$.

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